

Laser interferometric detectors of gravitational waves

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Abstract

A laser interferometric detector of gravitational waves is studied and a complete solution (to first order in the metric perturbation) of the coupled Einstein-Maxwell equations with appropriate boundary conditions for the light beams is determined. The phase shift, the light deflection and the rotation of the polarization axis induced by gravitational waves are computed. The results are compared with previous literature, and are shown to hold also for detectors which are large in comparison with the gravitational wavelength.

To appear in *Classical and Quantum Gravity*

1 Introduction

The detection of gravitational waves is an important goal of contemporary physics and in recent years, research in the field has shifted towards laser interferometric detectors ([1]-[4] and references therein). Large laser interferometers are presently under construction and in the planning stages in various countries. Remarkable technological progress has been made in the effort to detect the weak gravitational waves that are believed to be emitted from extragalactic sources.

The analysis of a laser interferometric detector is usually carried out in Fermi normal coordinates (FNC), which are associated with a freely falling observer carrying the detector [5, 1]. In order to introduce these coordinates, one must restrict oneself to considering detectors with size a much smaller than the reduced wavelength $\lambda/2\pi$ of the gravitational waves to be detected. This condition is satisfied only marginally in some cases [1], and as larger interferometers are being planned, it will be satisfied even less and ultimately, not at all in certain frequency bands. Most detectors are conceived to operate at gravitational wave frequencies $\nu_g \leq 10$ KHz, with maximum sensitivity up to 1 KHz. The Italian-French project VIRGO plans the construction of an interferometer with arm size $a \sim 3$ Km, and maximum sensitivity around 100 Hz. The detector in its final stage is expected to operate in the range from 10 Hz to 1 KHz. For $\nu_g \sim 1$ KHz, $2\pi a/\lambda_g \simeq 0.06$, and in the extreme case $\nu_g \sim 10$ KHz, $2\pi a/\lambda_g \simeq 0.6$. Beam detectors in space for the detection of low frequency gravitational waves ($\nu_g \sim 10^{-1}$ Hz) have also been proposed ([6, 7]; [1] and references therein), although the necessary technology is not available at present; for these systems, $a \sim 10^6$ Km and $\nu_g \sim 10^{-1}$ Hz give $2\pi a/\lambda_g \sim 2$. Therefore, it appears convenient to generalize known results to the case $a \geq \lambda_g/2\pi$. Moreover, since energy localization and transfer in general relativity are not in as nearly a well-established state as they are in the rest of physics, it is particularly important that the theoretical predictions for the effects of gravitational waves in their interaction with proposed detectors be established with great care. The advantage of laser interferometry as a detection mechanism is that it requires no reference to an energy transfer mechanism (or even whether such a transfer actually occurs [8]) but rather, can be phrased entirely in terms of phase shifts. In fact, this carries with it the additional advantage of being expressible in terms of scalar quantities upon which all observers must agree. In a recent paper, Lobo [9] usefully focussed upon these aspects.

The formalism for the interference between electromagnetic and gravitational waves was established many years ago [10] and the perturbations to the electromagnetic waves in the case of weak fields were solved for longitudinal and transverse orientations. Moreover, they were solved with reference to well-defined boundary conditions. In Lobo's work, the phase shift is calculated for a wave propagating in one direction and this is added to the phase shift for a wave travelling in the opposite direction. However, to correspond to the actual physical configuration of a laser interferometer, it is necessary to be more specific and analyze

the response to a gravity wave of electromagnetic waves which travel equal distances along the interferometer arms at right angles to each other and then reflect at the mirrors which terminate these arms. Only when this is done can one have confidence in the expected interference which one seeks as the waves are recombined. In this paper, we use the electromagnetically gauge-invariant perturbation formalism of Ref. [10] in conjunction with the boundary conditions appropriate to a laser interferometer, to derive the expected phase shift as the reflected waves recombine with interference. Using this formalism, we are also able to compute the extent of the deflection of the laser beams due to the interference with the gravitational waves. While this is a first order effect in the metric perturbation, it only induces a second order contribution to the phase shift. The value of the deflection angle is then compared with the results from previous analyses made in a different context. In addition, we show that gravitational waves do not induce any rotation in the polarization plane of electromagnetic radiation.

The plan of the paper is as follows: in Sec. 2 an idealized model of an interferometric detector is described, and the Maxwell equations in the presence of a weak, monochromatic, optimally polarized gravitational wave are solved for an example case. In Sec. 3, the solutions for the actual interferometer are derived from this case by means of suitable coordinate transformations, and the phase shift in the light coming from the two interferometer arms is computed. In Sec. 4, we take into account the deflection of light beams and changes in the polarization induced by gravitational waves, showing that these effects are negligible. Finally, the phase shift is compared with previous literature in Sec. 5.

2 An idealized beam detector

We consider a simplified model of a laser interferometric detector, whose geometry is given in Fig. 1. The detector consists of a Michelson interferometer, in which three masses are hanging in the plane $z = 0$. The periods of the suspension pendula are much greater than the gravitational wave period T_g , so the masses can be regarded as freely falling. A beam splitter at the origin splits the light from a laser lamp into two beams of equal intensity which travel along the x and y axes, and are reflected by mirrors placed at a distance a from the origin, and perpendicular to the x and y axes, respectively. The reflected beams are collected at a photodiode which compares them and detects any differential effect induced by an impinging gravitational wave in the interferometer arms.

We assume that (see Ref. [9] for a discussion of these assumptions)

- the electromagnetic field in the laser beam can be treated as a test field;
- the mirrors at $x = a$, $y = a$ are perfectly reflecting;
- the transverse dimension of the light beam is much smaller than the gravitational wavelength λ_g ;

- the light frequency is much larger than the gravitational wave frequency,

$$\omega \gg \omega_g . \quad (2.1)$$

We consider a monochromatic gravitational wave travelling along the positive z axis with a single polarization, and with polarization axes along the “L” of the interferometer. The spacetime metric is given by¹

$$g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu} \quad (2.2)$$

in an inertial frame of the Minkowskian background metric $\eta_{\mu\nu}$, and $|h_{\mu\nu}| \ll 1$. For a polarized beam in the TT gauge, the only nonvanishing $h_{\mu\nu}$ describing the gravitational wave are

$$h_{11} = -h_{22} = -A \cos(k_g z - \omega_g t) , \quad (2.3)$$

where A is a (small) constant and $k_g = \omega_g/c$. Although we will restrict our attention to this particular direction of propagation and to this polarization of the gravitational wave, the calculation can be generalized to arbitrary incidence and polarization, in the manner of Ref. [9].

The unperturbed electromagnetic field in the x arm of the interferometer has the only nonvanishing components

$$E_y^{(0)} = -F_{02}^{(0)} = E_0 \left[e^{i(kx - \omega t)} - e^{-i(kx + \omega t - 2ka)} \right] , \quad (2.4)$$

$$H_z^{(0)} = F_{12}^{(0)} = E_0 \left[e^{i(kx - \omega t)} + e^{-i(kx + \omega t - 2ka)} \right] , \quad (2.5)$$

where $E_0 = \text{constant}$, $k = \omega/c$, and where only the real components of the complex fields need to be retained. The unperturbed field in the y arm is given by

$$E_x^{(0)} = -F_{01}^{(0)} = -E_0 \left[e^{i(ky - \omega t)} - e^{-i(ky + \omega t - 2ka)} \right] , \quad (2.6)$$

$$H_z^{(0)} = F_{12}^{(0)} = E_0 \left[e^{i(ky - \omega t)} + e^{-i(ky + \omega t - 2ka)} \right] . \quad (2.7)$$

These fields satisfy the flat space Maxwell equations

$$F^{(0)\mu\nu}{}_{,\nu} = 0 , \quad (2.8)$$

$$F_{\mu\nu,\rho}^{(0)} + F_{\nu\rho,\mu}^{(0)} + F_{\rho\mu,\nu}^{(0)} = 0 , \quad (2.9)$$

with perfectly reflecting boundary conditions at the mirrors located at $(a, 0, 0)$ and $(0, a, 0)$ (see Fig. 1). We decompose the electromagnetic field tensor as the sum of the unperturbed field and a small perturbation induced by the gravitational wave

$$F_{\mu\nu} \equiv F_{\mu\nu}^{(0)} + F_{\mu\nu}^{(1)} , \quad (2.10)$$

¹The metric signature is +2, and c denotes the velocity of light in vacuum. Greek indices run from 0 to 3 and Latin indices run from 1 to 3. We will perform computations to first order in the metric perturbation $h_{\mu\nu}$.

with $|F_{\mu\nu}^{(1)}| \ll |F_{\mu\nu}^{(0)}|$. The field equations for the perturbations $F_{\mu\nu}^{(1)}$ in the TT gauge have been obtained in Ref. [10]:

$$F_{\mu\nu,\rho}^{(1)} \eta^{\rho\nu} = h_{\mu}{}^{\nu,\rho} F_{\nu\rho}^{(0)} + h^{\nu\rho} F_{\mu\nu,\rho}^{(0)} + O(h^2) , \quad (2.11)$$

$$F_{\mu\nu,\rho}^{(1)} + F_{\nu\rho,\mu}^{(1)} + F_{\rho\mu,\nu}^{(1)} = 0 . \quad (2.12)$$

It has been shown in Ref. [11] that other expressions in the literature are incorrect because they do not account properly for the Lorentz condition². Equations (2.11) and (2.12) must be solved in both arms using perfectly reflecting boundary conditions. We solve these equations in a special orientation which does not correspond to an actual interferometer arm in our geometry depicted in fig. 1. We then obtain the solution to Eqs. (2.11) and (2.12) for the actual interferometer arms by means of suitable coordinate transformations. The fictitious system is composed of an electromagnetic wave propagating along the z axis, from $z = 0$ to $z = a$, where it is reflected by a perfect mirror; the system is perturbed by a gravitational wave described by

$$h_{11} = -h_{33} = A \cos(k_g y - \omega_g t) . \quad (2.13)$$

The unperturbed electromagnetic field is given by

$$E_x^{(0)} = E_0 \left[e^{i(kz - \omega t)} - e^{-i(kz + \omega t - 2ka)} \right] , \quad (2.14)$$

$$H_y^{(0)} = E_0 \left[e^{i(kz - \omega t)} + e^{-i(kz + \omega t - 2ka)} \right] . \quad (2.15)$$

Equations (2.11) for the electromagnetic tensor perturbations are

$$F_{01,1}^{(1)} + F_{02,2}^{(1)} + F_{03,3}^{(1)} = 0 , \quad (2.16)$$

$$-F_{10,0}^{(1)} + F_{12,2}^{(1)} + F_{13,3}^{(1)} = h_{11,0} F_{01}^{(0)} - h_{11} F_{13,3}^{(0)} , \quad (2.17)$$

$$-F_{20,0}^{(1)} + F_{21,1}^{(1)} + F_{23,3}^{(1)} = 0 , \quad (2.18)$$

$$F_{30,0}^{(1)} + F_{31,1}^{(1)} + F_{32,2}^{(1)} = 0 , \quad (2.19)$$

from which we can easily derive an inhomogeneous wave equation for $F_{01}^{(1)}$:

$$\begin{aligned} F_{01,00}^{(1)} - F_{01,22}^{(1)} - F_{01,33}^{(1)} &= h_{11,00} F_{01}^{(0)} + h_{11,0} \left(F_{01,0}^{(0)} - F_{13,3}^{(0)} \right) - h_{11} F_{13,30}^{(0)} = \\ &= \frac{AE_0}{2} \left\{ (K^+)^2 \left[e^{i(kz + k_g y - \Omega^+ t)} - e^{-i(kz - k_g y + \Omega^+ t - 2ka)} \right] \right. \\ &\quad \left. + (K^-)^2 \left[e^{i(kz - k_g y - \Omega^- t)} - e^{-i(kz + k_g y + \Omega^- t - 2ka)} \right] \right\} , \end{aligned} \quad (2.20)$$

²Note that Eqs. (2.11), (2.17) and (2.20) are linear both in $h_{\mu\nu}$ and in $F_{\mu\nu}^{(0)}$, but not in these two kinds of variables simultaneously. As a consequence, one cannot use the complex representations for both fields at the same time, retaining only their real parts in the final results. However, one is always allowed to make use of the identity $\cos z = (e^{iz} + e^{-iz})/2$.

where

$$K^\pm \equiv k \pm k_g , \quad (2.21)$$

$$\Omega^\pm \equiv \omega \pm \omega_g . \quad (2.22)$$

The solution of Eqs. (2.16)-(2.19) and (2.12) with the perfectly reflecting boundary condition at $z = a$ and to first order in h is

$$F_{01}^{(1)} = -\frac{AE_0}{4kk_g} \left\{ (K^+)^2 \left[e^{i(kz+k_gy-\Omega^+t)} - e^{-i(kz-k_gy+\Omega^+t-2ka)} \right] - (K^-)^2 \left[e^{i(kz-k_gy-\Omega^-t)} - e^{-i(kz+k_gy+\Omega^-t-2ka)} \right] \right\} , \quad (2.23)$$

$$F_{12}^{(1)} = -\frac{AE_0}{4k} \left\{ K^+ \left[e^{i(kz+k_gy-\Omega^+t)} - e^{-i(kz-k_gy+\Omega^+t-2ka)} \right] + K^- \left[e^{i(kz-k_gy-\Omega^-t)} - e^{-i(kz+k_gy+\Omega^-t-2ka)} \right] \right\} , \quad (2.24)$$

$$F_{13}^{(1)} = -\frac{AE_0}{4k_g} \left\{ K^+ \left[e^{i(kz+k_gy-\Omega^+t)} + e^{-i(kz-k_gy+\Omega^+t-2ka)} \right] - K^- \left[e^{i(kz-k_gy-\Omega^-t)} + e^{-i(kz+k_gy+\Omega^-t-2ka)} \right] \right\} , \quad (2.25)$$

and the other $F_{\mu\nu}^{(1)} = 0$. We can now use this example case to write the solutions to Eqs. (2.11) and (2.12) for the actual interferometer arms.

3 Solutions of the Maxwell equations and computation of the phase shift

x arm:

By means of the coordinate transformation $\{x^\mu\} \mapsto \{x'^\mu\}$, with

$$\begin{aligned} ct' &= ct , \\ x' &= y , \\ y' &= z , \\ z' &= x , \end{aligned} \quad (3.1)$$

the unperturbed electromagnetic field tensor given by Eqs. (2.4) and (2.5) transforms in such a way that its only nonvanishing components in the $\{x'^\mu\}$ system are

$$E_x'^{(0)} = -F_{01}'^{(0)} = E_0 \left[e^{i(kz'-\omega t')} - e^{-i(kz'+\omega t'-2ka)} \right] , \quad (3.2)$$

$$H_y'^{(0)} = -F_{13}'^{(0)} = E_0 \left[e^{i(kz' - \omega t')} + e^{-i(kz' + \omega t' - 2ka)} \right] . \quad (3.3)$$

The metric perturbation becomes

$$h'_{11} = -h'_{33} = \frac{A}{2} \left[e^{i(k_g y' - \omega_g t')} + e^{-i(k_g y' - \omega_g t')} \right] . \quad (3.4)$$

Equations (3.2), (3.3) and (3.4) coincide with Eqs. (2.14), (2.15) and (2.13), the solutions of which are given by Eqs. (2.23)-(2.25). Transforming back to the $\{x^\mu\}$ system, these become

$$F_{02}^{(1)} = -E_y^{(1)} = -\frac{AE_0}{4kk_g} \left\{ (K^+)^2 \left[e^{i(kx + k_g z - \Omega^+ t)} - e^{-i(kx - k_g z + \Omega^+ t - 2ka)} \right] - (K^-)^2 \left[e^{i(kx - k_g z - \Omega^- t)} - e^{-i(kx + k_g z + \Omega^- t - 2ka)} \right] \right\} , \quad (3.5)$$

$$F_{12}^{(1)} = H_z^{(1)} = \frac{AE_0}{4k_g} \left\{ K^+ \left[e^{i(kx + k_g z - \Omega^+ t)} + e^{-i(kx - k_g z + \Omega^+ t - 2ka)} \right] - K^- \left[e^{i(kx - k_g z - \Omega^- t)} + e^{-i(kx + k_g z + \Omega^- t - 2ka)} \right] \right\} , \quad (3.6)$$

$$F_{23}^{(1)} = H_x^{(1)} = -\frac{AE_0}{4k} \left\{ K^+ \left[e^{i(kx + k_g z - \Omega^+ t)} - e^{-i(kx - k_g z + \Omega^+ t - 2ka)} \right] + K^- \left[e^{i(kx - k_g z - \Omega^- t)} - e^{-i(kx + k_g z + \Omega^- t - 2ka)} \right] \right\} , \quad (3.7)$$

and the other $F_{\mu\nu}^{(1)} = 0$.

y arm:

We now perform the coordinate transformation

$$\begin{aligned} ct' &= ct , \\ x' &= x , \\ y' &= z , \\ z' &= -y , \end{aligned} \quad (3.8)$$

which gives

$$E_x'^{(0)} = E_0 e^{2ika} \left[e^{i(kz' - \omega t')} - e^{-i(kz' + \omega t' - 2ka')} \right] , \quad (3.9)$$

$$H_y'^{(0)} = E_0 e^{2ika} \left[e^{i(kz' - \omega t')} + e^{-i(kz' + \omega t' - 2ka')} \right] , \quad (3.10)$$

where $a' \equiv -a$, and the boundary condition $E_x^{(0)}(y = a) = 0$ is transformed to $E_x'^{(0)}(z' = a') = 0$. In addition,

$$h'_{11} = -h'_{33} = -\frac{A}{2} \left[e^{i(k_g y' - \omega_g t')} + e^{-i(k_g y' - \omega_g t')} \right] . \quad (3.11)$$

Thus, we reproduce the case described by Eqs. (2.13)-(2.15), provided we make the following substitutions:

$$E_0 \mapsto E_0 e^{2ika} , \quad (3.12)$$

$$A \mapsto -A . \quad (3.13)$$

The electromagnetic field perturbations are then given by Eqs. (2.23)-(2.25), with these substitutions. Transforming back to the $\{x^\mu\}$ coordinate system, we find that the only nonvanishing $F_{\mu\nu}^{(1)}$ are

$$F_{01}^{(1)} = -E_x^{(1)} = -\frac{AE_0}{4kk_g} \left\{ (K^+)^2 \left[e^{i(ky+k_gz-\Omega^+t)} - e^{-i(ky-k_gz+\Omega^+t-2ka)} \right] - (K^-)^2 \left[e^{i(ky-k_gz-\Omega^-t)} - e^{-i(ky+k_gz+\Omega^-t-2ka)} \right] \right\} , \quad (3.14)$$

$$F_{12}^{(1)} = H_z^{(1)} = -\frac{AE_0}{4k_g} \left\{ K^+ \left[e^{i(ky+k_gz-\Omega^+t)} + e^{-i(ky-k_gz+\Omega^+t-2ka)} \right] - K^- \left[e^{i(ky-k_gz-\Omega^-t)} + e^{-i(ky+k_gz+\Omega^-t-2ka)} \right] \right\} , \quad (3.15)$$

$$F_{13}^{(1)} = -H_y^{(1)} = -\frac{AE_0}{4k} \left\{ K^+ \left[e^{i(ky+k_gz-\Omega^+t)} - e^{-i(ky-k_gz+\Omega^+t-2ka)} \right] + K^- \left[e^{i(ky-k_gz-\Omega^-t)} - e^{-i(ky+k_gz+\Omega^-t-2ka)} \right] \right\} . \quad (3.16)$$

In the TT gauge, the effect of the gravitational wave is to generate two sideband components of the electromagnetic signal at frequencies Ω^\pm , which propagate together with the carrier of frequency ω . This can be seen as a phase shift at the photodiode, where the two signals reflected by the mirrors are compared.

In the x arm, in the limit $\omega \gg \omega_g$, the backward propagating component of the electric field on the $z = 0$ plane is

$$\begin{aligned} E_{yb} = & -E_0 \cos(kx + \omega t - 2ka) - \frac{AE_0}{4} \frac{k}{k_g} [\cos(kx + \Omega^+t - 2ka) \\ & - \cos(kx + \Omega^-t - 2ka)] = -E_0 \cos(kx + \omega t - 2ka) \\ & + \frac{AE_0}{2} \frac{k}{k_g} \sin(kx + \omega t - 2ka) \sin(\omega_g t) . \end{aligned} \quad (3.17)$$

By setting

$$\begin{aligned} E_{yb} & \equiv -E_0 \cos(kx + \omega t - 2ka + \delta\phi_x) \\ & \simeq -E_0 \cos(kx + \omega t - 2ka) + E_0 \delta\phi_x \sin(kx + \omega t - 2ka) , \end{aligned} \quad (3.18)$$

and comparing Eqs. (3.17) and (3.18) at the arrival time τ of the signals at the photodiode (τ is the round trip time for the laser light in each of the interferometer arms), we find the phase shift for the x arm:

$$\delta\phi_x = \frac{A}{2} \frac{\omega}{\omega_g} \sin(\omega_g \tau) . \quad (3.19)$$

Analogously, we obtain the phase shift for the y arm

$$\delta\phi_y = -\frac{A}{2} \frac{\omega}{\omega_g} \sin(\omega_g \tau) , \quad (3.20)$$

which is equal and opposite to $\delta\phi_x$. The differential phase shift between the signals coming from the two interferometer arms is thus

$$\delta\phi \equiv \delta\phi_x - \delta\phi_y = 2A \frac{\omega}{\omega_g} \sin\left(\frac{\omega_g \tau}{2}\right) \cos\left(\frac{\omega_g \tau}{2}\right) , \quad (3.21)$$

expressed in a form which is convenient for the purposes of comparison.

4 Beam deflection and polarization shift

We now consider the deflection of the laser beams induced by their interaction with the gravitational waves. The fact that gravitational waves curve light rays propagating through them is well known, and has been studied in the geometric optics approximation by various authors, mainly in view of its possible astrophysical consequences [16]-[19]. To the best of our knowledge, it has not been discussed previously in the context of interferometry. In principle, it could be of significance because it implies a change in the distance of beam traversal and as a consequence, a variation in phase of the waves. However, we will demonstrate that this phase shift is negligible in comparison with that discussed in Sec. 3 under the normal conditions currently considered.

We now analyze the configuration described by Eqs. (2.13)-(2.15): The unperturbed Poynting vector of the electromagnetic waves becomes

$$\vec{S}^{(0)} = \frac{c}{4\pi} \vec{E}^{(0)} \times \vec{H}^{(0)} = \frac{c}{4\pi} E_x^{(0)} H_y^{(0)} \vec{e}_z \quad (4.1)$$

(where \vec{e}_i denotes the unit vector parallel to the i -th axis). In the presence of the gravitational wave, the Poynting vector is

$$\vec{S}^{(tot)} = \frac{c}{4\pi} \vec{E} \times \vec{H} = \vec{S}^{(0)} + \vec{S}^{(1)} , \quad (4.2)$$

where

$$\vec{S}^{(1)} = \frac{c}{4\pi} \left[\left(E_x^{(0)} H_y^{(1)} + E_x^{(1)} H_y^{(0)} \right) \vec{e}_z - E_x^{(0)} H_z^{(1)} \vec{e}_y \right] + O(2). \quad (4.3)$$

Clearly, the vector $\vec{S}^{(tot)}$ is rotated in the (y, z) plane with respect to $\vec{S}^{(0)}$. We consider, for the sake of simplicity, only the forward propagating component of the electromagnetic wave. Let θ be the angle between the vectors $\vec{S}_{forward}^{(0)}$ and $\vec{S}_{forward}^{(tot)}$ in the (y, z) plane where they lie, i.e. the deflection angle. We have

$$S_{y \text{ forward}}^{(tot)} = \left| \vec{S}_{forward}^{(tot)} \right| \sin \theta , \quad (4.4)$$

from which we obtain (we omit the subscript “forward” in the following)

$$\sin \theta = - \frac{H_z^{(1)}}{|H_y^{(0)}|} \text{sign}(E_x^{(0)}) + O(2) . \quad (4.5)$$

In the limit $\omega \gg \omega_g$, in the plane $y = 0$, and expanding $\sin \theta$ to first order, we find

$$\theta = \frac{A}{2} \cos(\omega_g t) . \quad (4.6)$$

Note that $\theta = O(h)$. Equation (4.6) can be compared with the deflection angle obtained in previous analyses performed in the geometric optics approximation. If we consider a photon whose unperturbed path is parallel to the z axis, and with four-momentum

$$p^\mu = p^{(0)\mu} + \delta p^\mu = (1, 0, 0, 1) + \delta p^\mu , \quad (4.7)$$

where δp^μ are small deflections, the equation of null geodesics give

$$\delta p^i = - \frac{1}{2} \int_S^O dz (h_{00} + 2h_{03} + h_{33})^i + O(2) \quad (4.8)$$

($i = 1, 2$), where the integral is computed along the unperturbed photon path from the light source S to the observer O . This formula can be found, e.g., in Refs. [17]-[20]. In our case we obtain, from Eq. (2.13),

$$\delta p^x = O(2) \quad (4.9)$$

$$\delta p^y = \frac{A}{2} \cos(\omega_g t) + O(2) , \quad (4.10)$$

which confirm that the deflection takes place in the (y, z) plane, and the value of θ is as in Eq. (4.6). This provides a check of our results from an independent source. It is of interest to consider the additional phase shift at the photodiode due to the change in path length travelled by the photons under the deflection. If δl_i is the extra length for the i -th arm ($i = 1, 2$), the phase shift it induces is $\delta \phi_i = 2\pi \delta l_i / \lambda$. It is immediate to see that, if l_i is the total length travelled by the photon,

$$\delta l_i = l_i - a = l_i(1 - \cos \theta) \simeq a \frac{\theta^2}{2} . \quad (4.11)$$

The additional phase shift introduced by the deflection of light is thus $\delta\phi_i = O(h^2)$, and is completely negligible. This is true also for the differential phase shift in the two arms of the interferometer.

An interesting property of the interaction between gravitational and electromagnetic waves can be seen from the solutions (2.23)-(2.25) of the coupled Einstein-Maxwell equations. We note that the unperturbed electric and magnetic fields are polarized along the x and y axes, respectively. The perturbation to the electric field is parallel to the x axis, while that of the magnetic field has components along both the y and z axis. This is consistent with the result [21] that a weak gravitational wave induces no rotation in the plane of polarization of electromagnetic radiation, to first order in h . In fact, let us consider a given point O on the light beam, at a given time, and the two planes \mathcal{P}_1 and \mathcal{P}_2 passing through O , which are defined by the pairs of vectors $(\vec{E}^{(0)}, \vec{H}^{(0)})$ and $(\vec{E}^{(tot)}, \vec{H}^{(tot)})$, respectively (see Fig. 2). The normals to these planes are parallel to the Poynting vectors $\vec{S}^{(0)}$ and $\vec{S}^{(tot)}$ respectively. \mathcal{P}_1 is associated with the forward propagating electromagnetic wave at O at the given time, in the unperturbed case (no gravitational waves) and \mathcal{P}_2 with that corresponding to the perturbed wave. \mathcal{P}_1 and \mathcal{P}_2 intersect along their common x axis. The plane \mathcal{P}_2 can be obtained by rotating \mathcal{P}_1 by an angle θ around the x axis. The projection of the magnetic field $H^{(tot)} \in \mathcal{P}_2$ on \mathcal{P}_1 lies on the direction of $H^{(0)}$ (i.e. on the y axis of \mathcal{P}_1). In this sense, the variation of the direction of the magnetic field due to the gravitational wave arises solely from the deflection of the light beam. No rotation of the electric and magnetic field, with respect to the unperturbed case, takes place in a plane orthogonal to the Poynting vector (in the sense of the background Minkowski metric). Although these considerations could be made more rigorous by using advanced concepts from differential geometry, it is unnecessary, since a general proof of the fact that gravitational waves do not rotate the polarization vector of electromagnetic waves propagating through them, to first order, can be found in Ref. [21].

5 Discussion and conclusion

In Sec. 3 we have presented a complete solution (to first order in h) to the Maxwell equations in the field of a weak gravitational wave, taking into account the boundary conditions. Our result in Eq. (3.21) disagrees with that by Lobo by the presence of the factor $\cos(\omega_g\tau/2)$. This factor is absent in Lobo's paper [9], but it does appear in a paper by Meers [12]. This is contrary to Lobo's claimed agreement with Meers. To reproduce Lobo's result in our calculations, it would require a time interval $(0, \tau/2)$ to be considered twice instead of correctly considering the two distinct time intervals $(0, \tau/2)$ and $(\tau/2, \tau)$. The former approach would give a factor $2\sin(\omega_g\tau/2)$ in our Eqs. (3.19) and (3.20), and corresponds to computing the phase shift for the forward propagating signal during the time interval $(0, \tau/2)$, and incorrectly doubling the result in order to obtain the phase shift for the reflected signal travelling from the mirror to

the photodiode during the time interval $(\tau/2, \tau)$. Apparently, the final formulas for the phase shift in Ref. [9] are obtained in this way. The correct computation gives a factor $\sin(\omega_g \tau) = 2 \sin(\omega_g \tau/2) \cos(\omega_g \tau/2)$ instead of $2 \sin(\omega_g \tau/2)$ in Eqs. (3.19) and (3.20). However, it should be noted that the distinction is irrelevant in the limit $\tau/T_g \ll 1$, wherein both Eq. (3.21) and the result of Ref. [9] agree with Ref. [1]. This limit is not unphysical: if $\tau/T_g \sim 1$, Eq. (3.21) gives $\delta\phi \sim 0$, and the detector is rendered useless due to the fact that the metric perturbation reverses its sign and the phase shift accumulated in the first half period of the gravitational wave is destroyed during the second half [1]. However, the inherent broadband nature of the detector makes it sensitive to gravitational wave pulses which are not monochromatic, but have Fourier components in a band of frequencies $\Delta\nu_g$, some of which may have $\omega_g \tau$ less than 1, but not very small. In such cases the full Eq. (3.21) applies.

Perhaps, the most promising sources of gravitational waves detectable in the near future with beam detectors are coalescing binaries containing compact objects. These systems give rise to a characteristic signal (“chirp”) with frequency increasing up to 1 KHz in the final stages of coalescence. The final moments of the process are not completely understood [13, 14, 15], and might well give rise to higher frequencies not satisfying $\omega_g \tau \ll 1$.

In Sec. 4, we used the formalism in Ref. [10] to deduce the deflection of the laser beams in their interaction with the gravitational waves. We found that, while this is a first order effect in h , it only induces a second order contribution to the phase shift.

Finally, we note that if perfectly reflecting mirrors are placed at $x = 0$, $y = 0$ normal to the x and y axes, the wave number of the unperturbed electromagnetic field in the cavity so constructed is restricted to the values $k = n\pi/a$, where n is an integer. This system can provide an idealized description of a laser interferometric detector operating with Fabry-Perot cavities in its arms ([1]-[4] and references therein).

Acknowledgments

V. F. acknowledges financial support from the Fondazione Angelo della Riccia and the warm hospitality at the University of Victoria. This research was supported, in part, by a grant from the Natural Sciences and Engineering Research Council of Canada.

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Figure captions:

Figure 1: The gravitational wave propagates along the positive z axis; electromagnetic waves starting from the beam splitter at the origin are reflected by mirrors placed at $x = a$, $y = a$.

Figure 2: The effects induced by the gravitational wave on the plane of polarization of a forward propagating electromagnetic wave. $\vec{E}^{(0)}$, $\vec{H}^{(0)}$ and $\vec{S}^{(0)}$ are the electric and magnetic

fields, and the Poynting vector, respectively, in the unperturbed case. $\vec{E}^{(tot)}$, $\vec{H}^{(tot)}$ and $\vec{S}^{(tot)}$ are the corresponding quantities in the presence of gravitational waves. θ is the deflection angle of the beam. The planes \mathcal{P}_1 and \mathcal{P}_2 are the polarization planes of the electromagnetic radiation in the unperturbed and perturbed case.